Similarly, we define $P^{(1)}$ to be the loading induced by a plunging downwash $\bar{x}$. Since P_α corresponds to $1+ikx = 1+ik(\bar{x}+1)/2$, we have

$$P_\alpha(x) = (1+ik/2)\bar{P}^{(0)}(\bar{x}) + ik\bar{P}^{(1)}(\bar{x})/2 \tag{6}$$

Therefore,

$$k^{-2}[(1+ik)P_h - ikP_\alpha] = k^{-2}[(1+ik)ik\bar{P}^{(0)} - ik(1+ik/2)\bar{P}^{(0)} - (ik)^2\bar{P}^{(1)}/2] = [\bar{P}^{(1)} - \bar{P}^{(0)}]/2 \tag{7}$$

Use of Eqs. (3, 4b, and 5) enable us to write Eq. (1) as

$$L = \frac{1}{2}\int_{-1}^{1} \bar{w}(-\bar{x})\bar{P}^{(0)}(\bar{x})\,\mathrm{d}\bar{x} \tag{8}$$

Similarly, Eqs. (3, 4b, and 7) transform Eq. (2) to

$$M = -\frac{1}{4}\int_{-1}^{1} \bar{w}(-\bar{x})[\bar{P}^{(1)}(\bar{x}) - \bar{P}^{(0)}(\bar{x})]\mathrm{d}\bar{x} = -\frac{1}{4}\int_{-1}^{1} \bar{w}(-\bar{x})\bar{P}^{(1)}(\bar{x})\,\mathrm{d}\bar{x} + \frac{1}{2}L \tag{9}$$

where Eq. (8) has been used in the last step. Finally, the nose-down moment about the center is

$$M_0 = M - \frac{1}{2}L = -\frac{1}{4}\int_{-1}^{1} \bar{w}(-\bar{x})\bar{P}^{(1)}(\bar{x})\,\mathrm{d}\bar{x} \tag{10}$$

Equations (8) and (10) are the present versions of Williams' results. The first gives the lift as an integral involving the loading induced by unit plunging downwash $\bar{P}^{(0)}$, as Williams found. The second gives the nose-down moment about the airfoil center as an integral involving only $\bar{P}^{(1)}$; the loading induced by a pitching downwash whose chordwise dependence is $\bar{x}$. In both cases, the arbitrary motion of the airfoil is represented by the downwash $\bar{w}(\bar{x})$, which appears in the integrals as $\bar{w}(-\bar{x})$. In the present centered coordinate system, then, not only are lift and moment for arbitrary motion determined only by the loadings for plunging and pitching, but the lift depends only on the plunging loading and the moment only on the pitching loading. This seems to be a useful reformulation of Williams' results.

We also note that for unit downwash, Eq. (10) gives

$$M_0^{(0)} = -\frac{1}{4}\int_{-1}^{1} \bar{P}^{(1)}(\bar{x})\,\mathrm{d}\bar{x} = -\frac{1}{2}L^{(1)} \tag{11}$$

by the definition of $\bar{P}^{(1)}$. This relation between the moment due to plunging and the lift due to pitching is well-known, and corresponds to Williams' more complicated Eq. (20).

It is easy to verify that Eqs. (8) and (10) give correct results when applied to the classical cases of single airfoils oscillating in incompressible flow for the various types of downwash. The same is true of in-phase oscillations of unstaggered airfoil cascades.[2]

References

[1] Williams, M. H., "Exact Solutions in Oscillating Airfoil Theory," *AIAA Journal*, Vol. 15, June 1977, pp. 875-877.

[2] Kemp, N. H. and Ohashi, H., "Forces on Unstaggered Airfoil Cascades in Unsteady In-Phase Motion," *Journal of Fluids Engineering*, Vol. 98, Sept. 1976, pp. 521-530.

Nonuniform Propagation of Sonic Discontinuities Through Thermally Conducting Gases

Rama Shankar* and Mohan Prasad†
Indian Institute of Technology Delhi, Hauz Khas, New Delhi, India

Introduction

SHOCK formation is a feature of ideal gasdynamics. The basic equations are hyperbolic and nonlinear. The motion of a medium is governed by a system of equations such that an initially smooth wave front may steepen to form a compressive shock. Using the theory of singular surfaces, which was proposed and developed by Thomas[1] to study the growth of discontinuities in the continuum mechanics, Thomas,[2] Kaul,[3] Nariboli,[4,5] and Shankar[6,7] have studied the propagation of weak discontinuities through different media, under the assumption that the medium in front of the propagating surface is uniform and at rest. If the medium ahead is moving, then it can be shown[8] that the wave propagation is anisotropic. In order to study anisotropic wave propagation, Lighthill[9] has developed an elegant method which essentially involves the evaluation of the Fourier integrals by the stationary phase method and gives the asymptotic features of the solution. Numerous applications of this method followed. Ludwig[10] and Duff[11] further generalized and developed this technique.

The mathematical theory of geometric optics (ray theory) of Luneberg[12] was found useful by Bazer[13] in the investigation of the propgation of weak discontinuities. Nariboli[14] combined the theory of singular surfaces and Luneberg's method of geometric optics to study the propagation of weak discontinuities in nonlinear anisotropic media. Using this combination, he integrated the equation of growth of discontinuities in a simple and straightforward manner. Following Nariboli,[14] Upadhyay[15] obtained the growth equation for sonic discontinuities propagating through thermally conducting gases, but he did not discuss the growing or decaying tendency.

Recently, Elcrat[16] studied the nonuniform propagation of sonic discontinuities in an unsteady flow of a perfect gas. In order to integrate the growth equations, he transformed them to an equation along the bicharacteristic curve in the characteristic manifold. While doing so, he arrived at an ordinary differential equation which was solved completely, and the ciriterion for decay or "blow up" was obtained. In the present Note, following Elcrat,[16] we shall derive and discuss the solutions of fundamental differential equations for nonuniform propagation of sonic discontinuities through the thermally conducting gases. We shall also find the criterion for decay or "blow up" for sonic discontinuities.

II. Inviscid Nonconducting Gas with Finite Thermal Conductivity

The set of nonlinear differential equations governing the inviscid nonconducting gas with finite thermal conductivity is given by[15],

$$\frac{\partial \rho}{\partial t} + v_i,\rho_i + \rho v_{i,i} = 0 \tag{1}$$

Received Feb. 28, 1978.

Index categories: Shock Waves and Detonations; Supersonic and Hypersonic Flow; Atmospheric and Space Sciences.

*Lecturer, Dept. of Mathematics.
†Research Scholar, Dept. of Mathematics.

$$\rho\left(\frac{\partial v_i}{\partial t}\right)+\rho v_j v_{i,j}+p_{,i}=0 \tag{2}$$

$$\rho\theta\left(\frac{\partial S}{\partial t}+v_i S_{,i}\right)=K\theta_{,ii} \tag{3}$$

where t denotes time, ρ the density, p the pressure, θ the temperature, v_i the components of flow velocity, and K the thermal conductivity. A repeated index implies summation, unless otherwise stated. The preceding equations refer to a system of rectangular coordinates, x^i, and a comma denotes the partial differentiation with respect to these coordinates in the usual manner.

The thermodynamic quantities are connected by

$$p=\rho R\theta \tag{4}$$

$$\theta \mathrm{d}S=Jc_v \mathrm{d}\theta-(p/\rho^2)\mathrm{d}\rho \tag{5}$$

where $R[=Jc_v(\gamma-1)]$ is the universal gas constant, J is the mechanical equivalent of heat, c_v is the specific heat at constant volume, and γ is the ratio of specific heats.

We consider a moving singularity surface $S(t)$ across which the field variables themselves are continuous, but their derivatives may be discontinuous.

Suppose that $S(t)$ is given by $F(x_i,t)=0$ and we denote by n_i the components of the unit normal vector $F_{,i}/|\mathrm{grad}\,F|$ and $G=-[(\partial F/\partial t)/|\mathrm{grad}\,F|]$. We consider the surface $S(t)$ as having two sides, denoted by ① and ②, and n_i points into ②. The relative speed of advancement of $S(t)$ in the fluid, $G-v_in_i$, is denoted by U. We use the square bracket to denote the jump, i.e., the difference in the value of a quantity behind and ahead. Let the discontinuities in the first derivatives be denoted by[16]

$$[v_{i,j}]=\lambda n_i n_j,\qquad \left[\frac{\partial v_i}{\partial t}\right]=-G\lambda n_i$$

$$[p_{,i}]=\xi n_i,\qquad \left[\frac{\partial p}{\partial t}\right]=-G\xi$$

$$[\rho_{,i}]=\zeta n_i,\qquad \left[\frac{\partial \rho}{\partial t}\right]=-G\zeta \tag{6}$$

where λ, ξ, and ζ are scalars defined on $S(t)$.

Using the first-order compatibility conditions in Eqs. (1) and (2), we get

$$U\zeta=\rho\lambda \tag{7}$$

and

$$\rho U\lambda=\xi=a^2\zeta \tag{8}$$

where a is the isothermal speed of sound, as given by $\sqrt{R\theta}$.

From Eqs. (7) and (8), we write

$$U^2=a^2 \tag{9}$$

which indicates that the sonic wave, in thermally conducting gaseous medium, is propagating with the isothermal velocity of sound.

Differentiating Eqs. (1) and (2) with respect to x_j, multiplying by n_j, summing with respect to j, and then taking the jump,[16] we get

$$U\frac{\delta\zeta}{\delta t}-(U^2\bar{\zeta}-\rho U\bar{\lambda}_i n_i)+2U\zeta(v_{i,j}n_in_j)_2$$
$$+2U\lambda\left[\left(\frac{\partial\rho}{\partial n}\right)_2-\rho\Omega\right]-2U\zeta\lambda+Ug^{\alpha\beta}v_i\zeta_{,\alpha}x_{i,\beta}=0 \tag{10}$$

and

$$\rho\frac{\delta\lambda}{\delta t}+(\bar{\xi}-U\rho\bar{\lambda}_in_i)-U\lambda\left(\frac{\partial\rho}{\partial n}\right)_2$$
$$+\left(\frac{\partial v_i}{\partial t}+v_kv_{i,k}\right)_2 n_i\zeta+U\zeta\left(\frac{\partial v_k}{\partial n}\right)_2 n_k$$
$$+U\zeta(v_{i,k}n_in_k)_2+\rho g^{\alpha\beta}v_k\lambda_{,\alpha}x_{k,\beta}=0 \tag{11}$$

where $\delta/\delta t$ is a differentiation along an orthogonal trajectory of the surface $S(t)$, which is given by $x_i=x_i(u_\alpha)$, $(\alpha=1,2)$; Ω is the mean curvature of $S(t)$, defined by $2\Omega=g^{\alpha\beta}b_{\alpha\beta}$; $g^{\alpha\beta}$ and $b_{\alpha\beta}$ are the first and second fundamental forms of $S(t)$, respectively; and remaining symbols have the same meaning as in Ref. 16.

From Eqs. (5) and (6), we get

$$\theta\left\{\left[\frac{\partial S}{\partial t}\right]+v_i[S_{,i}]\right\}=\frac{a^3}{\rho}\zeta \tag{12}$$

Now, taking the jump of Eq. (3) and using Eq. (12), we get

$$[\theta_{,ii}]=a^3\zeta/K \tag{13}$$

Differentiating the state equation (4), twice with respect to x_i, summing with respect to i and taking jumps, we get

$$\bar{\xi}=[p_{,ii}]=\rho R[\theta_{,ii}]+2R[\theta_{,i}\rho_{,i}]+R\theta[\rho_{,ii}] \tag{14a}$$

or,

$$\bar{\xi}=\rho Ra^3\zeta/K+2R(\theta_{,i})_2\zeta n_i+a^2\bar{\zeta} \tag{14b}$$

or

$$(\bar{\xi}-a^2\bar{\zeta})=\rho Ra^3\zeta/K+2R(\theta_{,i})_2\zeta n_i \tag{14c}$$

Now eliminating $\bar{\xi}$, $\bar{\lambda}_in_i$, and $\bar{\zeta}$ from Eqs. (10, 11, and 14), we get

$$U\left(\frac{\delta\zeta}{\delta t}+g^{\alpha\beta}v_i\zeta_{,\alpha}x_{i,\beta}\right)+\rho\left(\frac{\delta\lambda}{\delta t}+g^{\alpha\beta}v_k\lambda_{,\alpha}x_{k,\beta}\right)$$
$$+\frac{\rho Ra^3\zeta}{k}+2R(\theta_{,i})_2\zeta n_i+3U\zeta(v_{i,k}n_in_k)_2$$
$$+U\zeta\left(\frac{\partial v_k}{\partial n}\right)_2 n_k+U\lambda\left(\frac{\partial\rho}{\partial n}\right)_2-2U\rho\lambda\Omega-2U\zeta\lambda$$
$$+\left(\frac{\partial v_i}{\partial t}+v_kv_{i,k}\right)_2 n_i\zeta=0 \tag{15}$$

Equation (15) is a differential equation for $\zeta(U\zeta=\rho\lambda)$ and therefore, one for ξ and one for λ, along the orthogonal trajectories of $S(t)$. Since it is a Riccatti differential equation, it is amenable to analysis, at least in certain special cases. However, the "inhomogeneous terms" arising from the surface derivatives cause some difficulty in interpretation, and if we transform Eq. (15) into a differential equation along bicharacteristic curves in the characteristic manifold $\Sigma=\cup_t S(t)$, this difficulty disappears.

Following Elcrat,[16] we write

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t}=\frac{\delta\zeta}{\delta t}+g^{\alpha\beta}\zeta_{,\alpha}v_jx_{j,\beta}\qquad \frac{\mathrm{d}\xi}{\mathrm{d}t}=\frac{\delta\xi}{\delta t}+g^{\alpha\beta}\xi_{,\alpha}v_jx_{j,\beta}$$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t}=\frac{\delta\lambda}{\delta t}+g^{\alpha\beta}\lambda_{,\alpha}v_jx_{j,\beta} \tag{16}$$

where symbols have the same meaning as in Ref. 16. Now, using Eqs. (7) and (15) in Eq. (16), we get

$$\frac{d\lambda}{dt}+\lambda\left\{\frac{1}{2}\frac{d}{dt}\log\left(\frac{\rho}{U}\right)+\frac{1}{2\rho}P\right\}-\lambda^2=0 \tag{17}$$

where

$$P=\frac{\rho R^2 a^2}{K}+\frac{2\rho R}{U}(\theta,_i)_2 n_i+3\rho(v_{i,k}n_i n_k)_2+\rho\left(\frac{\partial v_k}{\partial n}\right)_2 n_k$$

$$+U\left(\frac{\partial\rho}{\partial n}\right)_2-2\rho U\Omega+\frac{\rho}{U}\left(\frac{\partial v_i}{\partial t}+v_k v_{i,k}\right)_2 n_i$$

Equation (17) is the basic differential equation for the growth and decay of weak discontinuities associated with the wave surface $S(t)$.

Integration of Eq. (17) yields

$$\lambda=\frac{\lambda_0\left(\frac{\rho}{\rho_0}\right)^{-1/2}\left(\frac{U}{U_0}\right)^{1/2}\exp\left[-\frac{1}{2\rho}\int_0^t P\mathrm{d}\tau\right]}{1-\frac{1}{2}\lambda_0\rho_0^{+1/2}U_0^{-1/2}\int_0^t\rho^{-1/2}U^{1/2}\exp\left[-\frac{1}{\rho}\int_0^t P\mathrm{d}\tau\right]\mathrm{d}\tau}$$

with $S(t)=S_0$, $\lambda=\lambda_0$, $\zeta=\zeta_0$, $\xi=\xi_0$, $\rho=\rho_0$ and $U=U_0$ at $t=0$.

If λ_0 is positive, we have the criteria

$$\int_0^T\rho^{-1/2}U^{1/2}\exp\left(-\frac{1}{\rho}\int_0^t P\mathrm{d}\tau\right)\mathrm{d}\tau=\frac{2U_0^{1/2}}{\lambda_0\rho_0^{1/2}}$$

for "blow up" at a finite time T.

If we associate $\lambda\to\infty$ with the formation of a shock, these remarks may be thought of as generalization of corresponding statement in Upadhyay.[15] Thus, the thermal conductivity allows the existence of a singular surface carrying a weak discontinuity in a nonuniform medium under consideration, which finally may grow into a shock—its role is to cause damping.

Acknowledgment

The authors are indebted to O. P. Bhutani for his valuable suggestions and constant encouragement in the presentation of this work.

References

[1] Thomas, T. Y., "Extended Compatibility Conditions for the Study of Surface of Discontinuity in Continuum Mechanics," *Journal of Mathematics and Mechanics*, Vol. 6, 1957, pp. 311-322.

[2] Thomas, T. Y., "The Growth and Decay of Sonic Discontinuities in Ideal Gases," *Journal of Mathematics and Mechanics*, Vol. 6, 1957, pp. 455-469.

[3] Kaul, C. N., "On the Singular Surface of Order One in Ideal Gases," *Journal of Mathematics and Mechanics*, Vol. 10, 1961, p. 393.

[4] Nariboli, G. A., "The Propagation and Growth of Sonic Discontinuities in Magnetohydrodynamics," *Journal of Mathematics and Mechanics*, Vol. 12, 1963, p. 141.

[5] Nariboli, G. A. and Secrest, B. G., "Weak Discontinuities in Magnetogasdynamics in the Presence of Dissipative Mechanisms," *Tensor (NS)*, Vol. 18(1), Jan. 1967, pp. 22-25.

[6] Shankar, R., "Singular Surface of Order One in Non-Equilibrium Dissociative Gasdynamics," *International Journal of Engineering Science*, Vol. 9, 1971, pp. 1157-1164.

[7] Shankar, R. and Chandran, P., "Growth and Decay of Sonic Discontinuities in Non-Equilibrium Gasdynamics," *Journal of Mathematical and Physical Sciences*, Vol. 11, June 1977, pp. 237-246.

[8] Landau, L. D. and Lifshitz, E. M., *Fluid Mechanics*, Pergamon Press, London, 1959, p. 259.

[9] Lighthill, M. J., "Studies in Magnetohydrodynamic Waves and Other Anisotropic Wave Motions," *Philosophical Transactions of the Royal Society of London*, Vol. 252A, 1960, p. 397.

[10] Ludwig, D. A., "The Singularities of the Riemann Function," A.E.C. Rept. NYO-9351, New York, 1961.

[11] Duff, G. F. D., "On Wave-Fronts and Boundary Waves," *Communications of Pure and Applied Mathematics*, Vol. 17, 1964, p. 189.

[12] Luneberg, R. K., *Mathematical Theory of Optic*, University of California Press, Berkeley, 1964.

[13] Bazer, J. and Fleischman, O., "Propagation of Weak Hydromagnetic Discontinuities," *The Physics of Fluids*, Vol. 2, 1959, pp. 366-378.

[14] Nariboli, G. A., Singh, S. N. and Ranga Rao, M. P., "Growth of Weak Discontinuties in Arbitrary Moving Gas," *Proceedings of the Indian Academy of Sciences*, Vol. 68(A), 1968, pp. 149-163.

[15] Upadhyay, K. S., "Propagation of Weak Discontinuity Through Thermally Conducting Gases," *Tensor (N.S.)*, Vol. 21, 1970, pp. 296-300.

[16] Elcrat, A. R., "On the Propagation of Sonic Discontinuities in the Unsteady Flow of a Perfect Gas," *International Journal of Engineering Sciences*, Vol. 15, 1977, pp. 29-35.

Type-Independent Solution for Mixed Compressible Flows

Wilson C. Chin*

Boeing Commercial Airplane Company, Seattle, Wash.

Introduction

THIS Note summarizes recent algorithm research for inviscid transonic flows. A parabolic potential equation is considered which consists of the usual inviscid equation plus a small high-order streamwise diffusion term. Relaxation schemes devised for the "viscous transonic equation" do not require mixed differencing, shock-point or sonic line operators. Also, convergence appears to be independent of sweep direction.

Discussion

The description of smooth flows is completely determined from the inviscid equations alone, but in the description of shocked flows, supplementary information on jump conditions must be supplied. This is necessary because high-order derivatives involving viscous terms have been omitted. That the Rankine-Hugoniot conditions are derivable from the full viscous equations[1] without appeal to external entropy conditions (the high-order system implicitly contains the required thermodynamic information) suggests that conservative differencing and special shock-point and sonic line operators might not be necessary, provided that we consider, instead of the low-order inviscid equation, an appropriate high-order problem.

Generally speaking, the physical details near sonic lines and shocks cannot be resolved using the inviscid equation, and one has to reconsider the circumstances under which high-order streamwise derivatives, multiplied by viscosity, become important. This limiting process was investigated by Sichel[2] for small-disturbance flows with insignificant vorticity generation, and the result is the "viscous transonic equation" (VTE) $\epsilon\varphi_{xxx}+(K-\varphi_x)\varphi_{xx}+\varphi_{yy}(x,y)=0$, where ϵ is proportional to an inverse Reynolds number based on the compressive viscosity, x and y are streamwise and normal coordinates, and K is the inviscid similarity parameter. The VTE is solved using $\varphi_y(x,0)=T'(x)$, $T'(x)$ being a nor-

Received Sept. 30, 1977; revision received April 27, 1978.

Index categories: Subsonic Flow; Transonic Flow.

*Specialist Engineer, Aerodynamics Research Group. Member AIAA.